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The Linear q-Difference Equation $y(x) = ay(qx) + f(x)$

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Abstract—The existence and uniqueness of solutions of the linear q-difference equation $y(x) = ay(qx) + f(x)$ in the function spaces $C[0,\infty)$ and $L^p(\mathcal{R}^+)$ are discussed. Numerical method via linear spline collocation is constructed and error estimates are given.

Keywords—Linear q-difference equation, Existence and uniqueness, Spline collocation method, Error estimate.

1. INTRODUCTION

In this paper, we consider the q-difference equation

$$y(x) = ay(qx) + f(x), \qquad x \geq 0, \tag{1}$$

where a is a complex constant, $q \in (0,1)$, and $f(x)$ is a given function. The motivation for our discussion is that the following system of equations:

$$y_i(x) = \sum_{j=1}^{n} a_{ij} y_j(q_{ij}x) + f_i(x), \qquad 1 \leq i \leq n, \quad x \geq 0 \tag{2}$$

arises in the study of the generalized Riemann problem for a quasilinear hyperbolic system of PDEs by Le Floch and Li [1] and of the Goursat problem for some linear hyperbolic systems of PDEs by Hua, Lin and Wu [2].

It is proved in [1] that if $\gamma := \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}| < 1$ and $f \in C^0(\mathcal{R}_+, \mathcal{R}^n)$, then (2) has one and only one solution in $C^0(\mathcal{R}_+, \mathcal{R}^n)$. One natural question is what happens if $\gamma \geq 1$. As far as (1) is concerned, we have the following result.

Theorem 1. *Suppose that $f(x) \in C[0,\infty)$ and consider the solubility of (1) in the function space $C[0,\infty)$.*

(1) *If $|a| > 1$ then there exists a one-to-one correspondence between solutions of (1) and functions in the space $C^* = \{z(x) \in C[q,1] \mid z(1) = az(q) + f(1)\}$.*
(2) *If $a = -1$, then (1) is solvable if and only if the series*

$$\sum_{k=0}^{\infty} (-1)^k \left(f\left(q^k x\right) - f(0)\right), \qquad x \in [0,\infty) \tag{3}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX *converges to a continuous function of* $x \in [0,\infty)$. *Subject to this condition, the only solution is* $y(x) = (1/2)f(0) + \sum_{k=0}^{\infty}(-1)^k(f(q^kx) - f(0))$.

(3) *If* $a = 1$, *then* (1) *is solvable if and only if* $f(0) = 0$ *and the series*

$$\sum_{k=0}^{\infty} f\left(q^k x\right), \qquad x \in [0,\infty) \tag{4}$$

converges to a continuous function of the $x \in [0,\infty)$. *Subject to this condition, all solutions can be written as* $y(x) = c + \sum_{k=0}^{\infty} f(q^kx)$, *where* c *is an arbitrary constant.*

It is easy to formulate examples of the continuous function $f(x)$ such that the series (3) or (4) diverges. However, a slightly greater restriction upon $f(x)$ such as the Hölder continuity at $x = 0$, i.e., $|f(x) - f(0)| \le Mx^\alpha$ in a neighbourhood of $x = 0$ for some positive constants M and $\alpha \in (0,1]$, is sufficient for the convergence of the series (3) or (4) to a continuous function of $x \in [0,\infty)$.

By transforming (2) into a system of difference equations, Serre [3] presented a necessary and sufficient condition of algebraic type for the existence and uniqueness of solutions in a Sobolev-type space (one may think of it as $L^2\left(\frac{dx}{x}\right)$). In this paper, we consider solutions in the more natural function space $L^p(\mathcal{R}^+)$, $1 \le p \le \infty$. As far as (1) is concerned, we have the following result.

THEOREM 2. *If* $q^{-1/p}|a| \ne 1$ *and* $f(x) \in L^p(\mathcal{R}^+)$, *then* (1) *has one and only one solution in* $L^p(\mathcal{R}^+)$.

The same result holds for equation (2) if $\max_{1\le i\le n} \sum_{j=1}^{n} q_{ij}^{-1/p}|a_{ij}| < 1$.

Finally, we discuss the numerical solution of (1). Let y_n be our approximation of the exact solution $y(x)$ at $x = nh$, $n = 0,1\dots$, where $h > 0$ is the grid length. Let $y^h(x)$ be the linear spline interpolation of $\{y_n\}_{n=0}^{\infty}$, i.e.,

$$y^h(x) = \frac{(n+1)h - x}{h} y_n + \frac{x - nh}{h} y_{n+1}, \qquad x \in [nh,(n+1)h), \quad n \ge 0.$$

By requiring that $y^h(x)$ satisfies (1) at $x = nh$, we obtain the recurrence relation

$$y_n = a y^h(qnh) + f_n, \qquad n \ge 0, \tag{5}$$

where $f_n = f(nh)$. For $t > 0$, we denote

$$\omega\left[g|_{[0,t]}, h\right] = \sup_{0\le x_1,x_2\le t,\ |x_1-x_2|\le h} |g(x_1) - g(x_2)|, \qquad \|g|_{[0,t]}\|_\infty = \max_{0\le x\le t} |g(x)|$$

for a function $g(x) \in C[0,\infty)$.

THEOREM 3. *If* $|a| < 1$, *then* (5) *has a unique solution* $\{y_n\}_{n=0}^{\infty}$ *which satisfies the error estimates*

$$|y_n - y(nh)| \le \begin{cases} \dfrac{2|a|}{(1-|a|)^2}\,\omega\left[f|_{[0,qnh]}, h,\right], & \text{if } f(x) \in C[0,\infty), \\ \dfrac{|a|h}{4(1-|a|)(1-q|a|)}\,\omega\left[f'|_{[0,qnh]}, h\right], & \text{if } f(x) \in C^1[0,\infty), \\ \dfrac{|a|h^2}{8(1-|a|)(1-q^2|a|)}\left\|f''|_{[0,qnh]}\right\|_\infty, & \text{if } f(x) \in C^k[0,\infty), \quad k \ge 2. \end{cases}$$

It is only a matter of notation to generalize the numerical method to (2) and to obtain similar error estimates under the condition that $\max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}| < 1$.

2. PROOFS OF THE THEOREMS

PROOF OF THEOREM 1. The last two parts of this theorem are trivial to prove. Let us consider the first part. For every solution $y(x) \in C[0,\infty)$ of (1), there exists a function $z(x) = y(x)$, $t \in [q,1]$, which is in C^*. On the other hand, for every function $z(x) \in C^*$, we can construct a solution $y(x)$ of (1) by the recurrence relation

$$\begin{aligned}
&y(0) = \frac{1}{1-a} f(0), && \\
&y(x) = z(x), && x \in [q,1], \\
&y(x) = a^{-1}\left(y\left(q^{-1}x\right) - f\left(q^{-1}x\right)\right), && x \in [q^{k+1}, q^k), \quad k = 1,2,\dots, \\
&y(x) = ay(qx) + f(x), && x \in (q^{-k}, q^{-k-1}], \quad k = 0,1,\dots.
\end{aligned}$$

The continuity of $y(x)$ at $x = 0$ can be justified by the relation

$$y\left(q^m x\right) = a^{-m} z(x) - \sum_{k=0}^{m-1} a^{k-m} f\left(q^k x\right), \qquad x \in [q,1], \quad m \geq 1. \qquad \blacksquare$$

PROOF OF THEOREM 2. We only consider the case where $p \in (1,\infty)$. The other two cases, namely $p = 1$ and $p = \infty$, can be discussed similarly. First, we consider the case where $q^{-1/p}|a| < 1$. Since $\|y(q\,\cdot)\|_p \leq q^{-1/p}\|y\|$ for all function $y(x) \in L^p(\mathcal{R}^+)$, we derive from the Contraction Mapping Theorem that (1) has one and only one solution in $L^p(\mathcal{R}^+)$. Next, we consider the case where $q^{-1/p}|a| > 1$. Applying the Laplace transform to (1), we obtain formally that

$$\hat{y}(t) = q^{-1} a \hat{y}\left(q^{-1}t\right) + \hat{f}(t), \qquad \operatorname{Re} t > 0,$$

where $\hat{y}(t) = \int_0^\infty e^{-xt} y(x)\, dx$ and $\hat{f}(t) = \int_0^\infty e^{-xt} f(x)\, dx$ are Laplace transforms of $y(x)$ and $f(x)$, respectively. Therefore,

$$\hat{y}(t) = qa^{-1}\hat{y}(qt) - qa^{-1}\hat{f}(qt), \qquad \operatorname{Re} t > 0.$$

By iteration, we have formally that

$$\hat{y}(t) = -\sum_{k=0}^{\infty} \left(qa^{-1}\right)^k \hat{f}\left(q^k t\right), \qquad \operatorname{Re} t > 0$$

which implies that

$$y(x) = -\sum_{k=0}^{\infty} \left(a^{-1}\right)^k f\left(q^{-k} x\right), \qquad x > 0.$$

It is easy to verify that this function is a solution of (1) in $L^p(\mathcal{R}^+)$. To prove the uniqueness of the solution, we only need to prove that when $f(t) \equiv 0$, equation (1) has the trivial solution only. Let $y(t) \in L^p(\mathcal{R}^+)$ be a solution of (1) with $f(t) \equiv 0$. Applying the Laplace transform to (1), we obtain

$$\hat{y}(t) = aq^{-1}\hat{y}\left(q^{-1}t\right), \qquad \operatorname{Re} t > 0,$$

which implies that

$$\hat{y}(t) = \left(a^{-1}q\right)^n \hat{y}\left(q^n t\right), \qquad n > 0, \quad \operatorname{Re} t > 0.$$

Noting that $|\hat{y}(t)| \leq (p'|t|)^{-1/p'}\|y\|_p$, where $p' \in (0,\infty)$ obeying $1/p + 1/p' = 1$, we have

$$|\hat{y}(t)| \leq \left(q^{-1/p}|a|\right)^{-n} (p'|t|)^{-1/p'}\|y\|_p, \qquad n > 0, \quad \operatorname{Re} t > 0.$$

By letting $n \to \infty$ in the preceding inequality, we see that $\hat{y}(t) = 0$ for all t with $\mathrm{Re}\, t > 0$. Therefore, $y(t) = 0$ a.e. in $L^p(\mathcal{R}^+)$. This proves that (1) has at most one solution in $L^p(\mathcal{R}^+)$. ∎

PROOF OF THEOREM 3. It is easy to see that (5) has a unique solution $\{y_n\}_{n=0}^{\infty}$. Let $m_n = [qn]$, $\delta_n = qn - [qn]$ and $f_n = f(nh)$, where $[\,\cdot\,]$ denotes the integer part. Then

$$y^h(qnh) = (1-\delta_n)y_{m_n} + \delta_n y_{m_n+1}.$$

Let $e_n = y_n - y(nh)$, $n \geq 0$. It follows from (1) and (5) that

$$e_n = a\left\{(1-\delta_n)e_{m_n} + \delta_n e_{m_n+1}\right\} + aE_n, \qquad n \geq 0,$$

where $E_n = (1-\delta_n)y(m_nh) + \delta_n y((m_n+1)h) - y(qnh)$. Hence,

$$|e_n| \leq |a| \max\left\{|e_{m_n}|, |e_{m_n+1}|\right\} + |aE_n|, \qquad n \geq 0.$$

It is easy to prove by induction from the preceding inequality that

$$|e_n| \leq \frac{|a|}{1-|a|} \max_{0\leq k\leq n} |E_k|, \qquad n \geq 0.$$

Our error estimates follow from the preceding inequality and the fact that

$$|E_n| \leq \begin{cases} 2\omega\left[y|_{[0,qnh]}, h\right], & \text{if } y(x) \in C[0,\infty), \\ \dfrac{h}{4}\,\omega\left[y'|_{[0,qnh]}, h\right], & \text{if } y(x) \in C^1[0,\infty), \\ \dfrac{h^2}{8}\left\|y''|_{[0,qnh]}\right\|_\infty, & \text{if } y(x) \in C^k[0,\infty), \quad k \geq 2, \end{cases}$$

for all $n \geq 0$, and that

$$\begin{aligned} \omega\left[y^{(k)}|_{[0,x]}, h\right] &\leq \frac{1}{1-q^k|a|}\,\omega\left[f^{(k)}|_{[0,x]}, h\right], && \text{if } f(x) \in C^k[0,\infty), \quad k \geq 0, \\ \|y''\|_{[0,x]} &\leq \frac{1}{1-q^2|a|}\left\|f''|_{[0,qnh]}\right\|_\infty, && \text{if } f(x) \in C^2[0,\infty), \end{aligned}$$

for all $x \geq 0$. ∎

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